Applied Mathematical Modelling 37 (2013) 172–186

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Effect of rotation on the onset of double diffusive convection in a Darcy porous medium saturated with a couple stress fluid

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ARTICLE INFO

Article history:
Received 12 August 2011
Accepted 9 February 2012
Available online 18 February 2012

Keywords:
Couple stress fluid
Rotation
Porous media
Double diffusive convection
Weak nonlinear
Heat mass transfer

ABSTRACT

The effect of rotation on the onset of double diffusive convection in a horizontal couple stress fluid-saturated porous layer, which is heated and salted from below, is studied analytically using both linear and weak nonlinear stability analyses. The extended Darcy model, which includes the time derivative and Coriolis terms, has been employed in the momentum equation. The onset criterion for stationary, oscillatory and finite amplitude convection is derived analytically. The effect of Taylor number, couple stress parameter, solute Rayleigh number, Lewis number, Darcy–Prandtl number, and normalized porosity on the stationary, oscillatory, and finite amplitude convection is shown graphically. It is found that the rotation, couple stress parameter and solute Rayleigh number have stabilizing effect on the stationary, oscillatory, and finite amplitude convection. The Lewis number has a stabilizing effect in the case of stationary and finite amplitude modes, with a destabilizing effect in the case of oscillatory convection. The Darcy–Prandtl number and normalized porosity advances the onset of oscillatory convection. A weak nonlinear theory based on the truncated representation of Fourier series method is used to find the finite amplitude Rayleigh number and heat and mass transfer. The transient behavior of the Nusselt number and Sherwood number is investigated by solving the finite amplitude equations using Runge–Kutta method.

1. Introduction

The problem of double diffusive convection in porous media has attracted considerable interest during the last few decades because of its wide range of applications, from the solidification of binary mixtures to the migration of solutes in water-saturated soils. Other examples include geophysical systems, electro-chemistry, and the migration of moisture through air contained in fibrous insulation. The problem of double diffusive convection in a porous medium has been extensively investigated and the growing volume of work devoted to this area is well documented by Ingham and Pop [1,2], Nield and Bejan [3], Vafai [4,5] and Vadasz [6].

Although the problem of double diffusive convection has been extensively investigated for Newtonian fluids, relatively little attention has been devoted to this problem with non-Newtonian fluids. The corresponding problem in the case of a porous medium has also not received much attention until recently. With growing importance of non-Newtonian fluids with suspended particles in modern technology and industries, the investigations of such fluids are desirable. The study of such fluids have applications in a number of processes that occur in industry, such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubrication and colloidal and suspension solutions. These fluids

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0307-904X/$ - see front matter
doi:10.1016/j.apm.2012.02.024

deform and produce a spin field due to the microrotation of suspended particles forming micropolar fluid developed by Eringen [7]. The micropolar fluids take care of local effects arising from microstructure and as well as the intrinsic motions of microfluidics. The spin field due to microrotation of freely suspended particles set up an anti-symmetric stress, known as couple stress, and thus forming couple stress fluid. Thus, couple stress fluid, according to Eringen [7], is a particular case of micropolar fluid when microrotation balances with the natural vorticity of the fluid. In the category of non-Newtonian fluids couple stress fluids have distinct features, such as polar effects. The theory of polar fluids and related theories are models for fluids whose microstructure is mechanically significant. The constitutive equations for couple stress fluids were given by Stokes [8]. The theory proposed by Stokes is the simplest one for micro-fluids, which allows polar effects such as the presence of couple stress, body couple and non-symmetric tensors. There are few studies available on the Rayleigh–Benard problem for couple stress fluids, and extensions including the issue of stability/onset [9–19].

The study of the effect of external rotation on thermal convection has attracted significant experimental and theoretical interest. Because of its general occurrence in geophysical and oceanic flows, it is important to understand how the Coriolis force influences the structure and transport properties of thermal convection. Rotating thermal convection also provides a system to study hydrodynamic instabilities, pattern formation and spatiotemporal chaos in nonlinear dynamical systems. The study of thermal convection in rotating porous media is motivated both theoretically and by its practical applications in engineering some of the important areas of applications in engineering include the food processing, chemical process, solidification and centrifugal casting of metals and rotating machinery. During the last two decades there has been a great deal of effort lead by many researchers on the study of effect of external rotation on the Rayleigh–Benard convection. In the literature there are plenty of works available on the problem of understanding how the Coriolis force influences the onset of thermal convection. The linear dynamics of rotating Rayleigh–Benard convection with rigid stress-free boundaries has been thoroughly investigated by Chandrasekhar [20] who determined the marginal stability boundary and critical horizontal wave numbers for the onset of convection and over stability as a function of the Taylor number. Vadasz [21] has used linear and weak nonlinear stability theories to study the effect of Coriolis force on gravity-driven convection in a rotating porous layer heated from below by employing the modified Darcy model. The differences as well as similarities between the porous medium and pure fluids convection results are highlighted in this study. An excellent review of research on thermal convection in a rotating porous medium is given by Vadasz [22]. A nonlinear stability analysis for thermal convection in a rotating porous layer has been performed by Straughan [23]. Chakrabarti and Gupta [24] have analyzed the nonlinear thermohaline convection in a rotating porous medium. Govender [25] has analyzed the effect of Coriolis force on centrifugally driven convection in a rotating layer of porous medium. Malashetty et al. [26] have studied linear and nonlinear thermal convection in a rotating porous layer using a thermal nonequillibrium model. Shivakumara et al. [27] have investigated the effect of Coriolis force on thermal convection in a layer of Newtonian fluid-saturated porous medium using the Brinkman–Lapwood–Darcy model with fluid viscosity different from Brinkman viscosity. The effect of rotation on the onset of double diffusive convection in a horizontal anisotropic porous layer was studied by Malashetty and Heera [28]. Convective instability in either a couple stress fluid layer or couple stress fluid-saturated porous layer heated from below has been investigated in the recent past including the effects of an additional diffusing component (i.e., solute concentration) and external constraints such as magnetic field and /or rotation.

Goel et al. [29] have studied the hydromagnetic stability of an unbounded couple stress binary fluid mixture under rotation with vertical temperature and solute concentration gradients. A layer of couple stress fluid saturating a porous medium heated from below in the presence of rotation has been studied by Sharma et al. [30], and condition for the onset of convection is obtained. Sunil et al. [31] have investigated the effect of magnetic field and rotation on a layer of couple stress fluid heated from below in a porous medium. Sharma and Sharma [32] have investigated the effect of suspended particles on electrically conducting couple stress fluid heated uniformly from below under the influence of uniform rotation and magnetic field. Recently, Shivakumara et al. [33] has discussed the Coriolis effect on thermal convection in a couple stress fluid-saturated rotating rigid porous layer. Further, the effect of rotation on the onset of double diffusive convection in a couple stress fluid-saturated porous medium is not available. The intent of the present paper is therefore to study the onset of double diffusive convection in a couple stress fluid-saturated rotating porous layer heated and salted from below using linear and weak nonlinear analyses with emphasis on how the condition for the onset of convection is modified in the presence of rotation and couple stresses.

2. Mathematical formulation

We consider an infinite horizontal fluid-saturated porous layer confined between the planes $z = 0$ and $z = d$, with the vertically downward gravity force $\mathbf{g}$ acting on it. The temperatures T_l and T_u with $T_l > T_u$ and solute concentrations S_l and S_u with $S_l > S_u$ are imposed at the bottom and top boundaries, respectively. The boundaries are impermeable, and we assume that the fluid and solid phases are in local thermal equilibrium. A Cartesian frame of reference is chosen with the origin in the lower boundary and the z-axis vertically upwards. The porous layer rotates uniformly about the z-axis with a constant angular velocity $\mathbf{\Omega} = (0, 0, \Omega)$. The interaction between heat and mass transfer, known as Soret and Dufour effects, is supposed to have no influence on the convective flow, so they are ignored. The velocities are assumed to be small so that the advective and Forchheimer inertia effects are ignored. The flow in the porous medium is governed by the modified Darcy's law, which includes the time derivative and the Coriolis terms is employed as a momentum equation.

The basic state is assumed to be quiescent, and we superpose infinitesimal perturbations on this basic state. The equations for the perturbation quantities under the Boussinesq approximation are

$$\nabla \cdot \mathbf{q} = 0, \tag{2.1}$$

$$\frac{\rho_0}{\varepsilon}\frac{\partial \mathbf{q}}{\partial t} = -\nabla p + \rho_0(\beta_T T - \beta_S S)\mathbf{g} - \frac{2\rho_0}{\varepsilon}\mathbf{\Omega} \times \mathbf{q} - \frac{1}{k}(\mu - \mu_c \nabla^2)\mathbf{q}, \tag{2.2}$$

$$\gamma \frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla)T = \kappa_T \nabla^2 T, \tag{2.3}$$

$$\varepsilon \frac{\partial S}{\partial t} + (\mathbf{q} \cdot \nabla)S = \kappa_S \nabla^2 S, \tag{2.4}$$

where $\mathbf{q} = (u, v, w)$ is the velocity, p is the pressure, ρ_0 is the reference density, $\mathbf{g}$ is the acceleration due to gravity, ε and k are the porosity and permeability of the porous medium, μ is the fluid viscosity, μ_c is the couple-stress viscosity, $\mathbf{\Omega}$ denote the angular velocity of rotation, T and S are the temperature and concentration, respectively, κ_T is the thermal diffusivity, κ_S is the solute diffusivity, β_T and β_S are the coefficients of thermal and solute expansion, respectively. Further, $\gamma = (\rho c)_m/(\rho c_p)_f$, where $(\rho c_p)_f$ is the volumetric heat capacity of the fluid and $(\rho c)_m = (1 - \varepsilon)(\rho c)_s + \varepsilon(\rho c_p)_f$ is the volumetric heat capacity of the saturated medium as a whole, with the subscripts f, s and m denoting the properties of the fluid, solid, and porous matrix, respectively. Eq. (2.2) is indeed the averaged equation applicable for couple stress fluid flow through rotating porous media. It is observed that the presence of rotation introduces an additional body force known as Coriolis force which has a profound effect on the flow of couple stress fluid through porous media. By operating curl twice on Eq. (2.2) we eliminate p from it, and then render the resulting equation and the Eqs. (2.3) and (2.4) dimensionless using the following transformations

$$\begin{aligned}&(x, y, z) = d(x^*, y^*, z^*), t = (\gamma d^2/\kappa_T)t^*, (u, v, w) = (\kappa_T/d)(u^*, v^*, w^*),\\ &T = (\Delta T)T^*, \quad S = (\Delta S)S^*\end{aligned} \tag{2.5}$$

to obtain non-dimensional equations as (on dropping the asterisks for simplicity),

$$\left[\left(\frac{1}{Pr_D}\frac{\partial}{\partial t} + 1 - C\nabla^2\right)^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right]w = \left(\frac{1}{Pr_D}\frac{\partial}{\partial t} + 1 - C\nabla^2\right)\nabla_1^2(Ra_T - Ra_S), \tag{2.6}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2 + \mathbf{q} \cdot \nabla\right]T - w = 0, \tag{2.7}$$

$$\left[\chi\frac{\partial}{\partial t} - Le^{-1}\nabla^2 + \mathbf{q} \cdot \nabla\right]S - w = 0, \tag{2.8}$$

where,

$$Ta = \left(\frac{2\Omega k}{\nu\varepsilon}\right)^2, \quad Pr_D = \frac{\varepsilon\gamma\nu d^2}{\kappa_T k}, \quad Ra_T = \frac{\beta_T g \Delta T d k}{\nu\kappa_T}, \quad Ra_S = \frac{\beta_S g \Delta S d k}{\nu\kappa_T}, \quad C = \frac{\mu_c}{\mu d^2}, \quad Le = \frac{\kappa_T}{\kappa_S}, \quad \chi = \frac{\varepsilon}{\gamma}.$$

The dimensionless groups, which appear, are the Taylor number Ta, Darcy–Prandtl number Pr_D, thermal Rayleigh number Ra_T, solute Rayleigh number Ra_S, couple stress parameter C, the Lewis number Le, and the normalized porosity χ.

Eqs. (2.6)–(2.8) are to be solved for stress free, isothermal, and isohaline boundaries. Hence the boundary conditions for the perturbation variables are given by

$$w = \frac{\partial^2 w}{\partial z^2} = T = S = 0, \quad \text{at } z = 0, 1. \tag{2.9}$$

3. Linear stability analysis

In this section we predict the thresholds of both marginal and oscillatory convections using linear theory. The eigenvalue problem defined by Eqs. (2.6)–(2.8) subject to the boundary conditions (2.9) is solved using the time-dependent periodic disturbances in a horizontal plane. Assuming that the amplitudes of the perturbations are very small, we write

$$(w, T, S) = (W(z), \Theta(z), \Phi(z))exp[i(lx + my) + \sigma t], \tag{3.1}$$

where l, m are horizontal wavenumbers and σ is the growth rate. Infinitesimal perturbations of the rest state may either damp or grow depending on the value of the parameter σ. Substituting Eq. (3.1) into the linearized version of Eqs. (2.6)–(2.8) we obtain

$$\left[\left(\frac{\sigma}{Pr_D} - C(D^2 - a^2) + 1\right)^2 (D^2 - a^2) + TaD^2\right]W + a^2 Ra_T\left(\frac{\sigma}{Pr_D} - C(D^2 - a^2) + 1\right)\Theta - a^2 Ra_S\left(\frac{\sigma}{Pr_D} - C(D^2 - a^2) + 1\right)\Phi = 0, \tag{3.2}$$

$$[\sigma - (D^2 - a^2)]\Theta - W = 0, \tag{3.3}$$

$$[\chi\sigma - Le^{-1}(D^2 - a^2)]\Phi - W = 0, \tag{3.4}$$

where $D \equiv d/dz$ and $a^2 = l^2 + m^2$. The boundary conditions (2.9) now read

$$W = D^2W = \Theta = \Phi = 0 \quad \text{at } z = 0, 1. \tag{3.5}$$

We assume the solutions of Eqs. (3.2)–(3.4) satisfying the boundary conditions (3.5) in the from

$$(W(z), \Theta(z), \Phi(z)) = (W_0, \Theta_0, \Phi_0) \sin n\pi z, \quad (n = 1, 2, 3, \ldots). \tag{3.6}$$

The most unstable mode corresponds to $n = 1$ (fundamental mode). Therefore, substituting Eq. (3.6) with $n = 1$ into Eqs. (3.2)–(3.4), we obtain a matrix equation

$$Ra_T = \frac{(\sigma + \delta^2)}{a^2}\left(\delta^2\left(\frac{\sigma}{\mathrm{Pr}_D} + \eta\right) + \frac{\pi^2 Ta}{\sigma Pr_D^{-1} + \eta} + \frac{a^2 Ra_S}{\chi\sigma + \delta^2 Le^{-1}}\right). \tag{3.7}$$

where $\eta = 1 + C\delta^2$ and $\delta^2 = \pi^2 + a^2$. We note that η is a representative of the viscosity of the fluid, and it is evident that the suspended particles add to the viscosity. The growth rate σ is in general a complex quantity such that $\sigma = \sigma_r + i\sigma_i$. The system with $\sigma_r < 0$ is always stable, while for $\sigma_r > 0$ it will become unstable. For neutral stability state $\sigma_r = 0$.

3.1. Stationary state

For the validity of principle of exchange of stabilities (i.e., steady case), we have $\sigma = 0$ (i.e.,$\sigma_r = \sigma_i = 0$) at the margin of stability. Then the Rayleigh number at which marginally stable steady mode exists becomes

$$Ra_T^{St} = \frac{(\pi^2 + a^2)^2(1 + C(\pi^2 + a^2))}{a^2} + \frac{Ta\pi^2(\pi^2 + a^2)}{a^2(1 + C(\pi^2 + a^2))} + LeRa_S. \tag{3.8}$$

The minimum value of the Rayleigh number Ra_T^{St} occurs at the critical wavenumber $a = a_c^{St}$ where $a_c^{St} = \sqrt{s}$ satisfies the equation

$$a_0 s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 - a_4 s - a_5 = 0, \tag{3.9}$$

where

$$\begin{aligned} &a_0 = 2C^3, \quad a_1 = 7\pi^2 C^3 + 5C^2, \quad a_2 = 4C + 12C^2\pi^2 + 8C^3\pi^4, \\ &a_3 = 1 + 5C\pi^2 + 6C^2\pi^4 - 4C\pi^4 Ta, \quad a_4 = 2C\pi^4 + 4C^2\pi^6 + 2C^3\pi^8 + 2C\pi^4 Ta, \\ &a_5 = \pi^4 + 3C\pi^6 + 3C^2\pi^8 + C^3\pi^{10} + \pi^4 Ta + C\pi^6 Ta. \end{aligned}$$

It is important to note that the critical wavenumber a_c^{St} depends on the couple stress parameter and Taylor number. In the absence of Taylor number, i.e. when $Ta = 0$, Eq. (3.8) gives

$$Ra_T^{St} = \frac{1}{a^2}(\pi^2 + a^2)^2[1 + C(\pi^2 + a^2)] + LeRa_S, \tag{3.10}$$

which is the result given by Malashetty et al. [16]. For single component fluid, $Ra_S = 0$, in Eq. (3.8) gives

$$Ra_T^{St} = \frac{(\pi^2 + a^2)^2(1 + C(\pi^2 + a^2))}{a^2} + \frac{Ta\pi^2(\pi^2 + a^2)}{a^2(1 + C(\pi^2 + a^2))}. \tag{3.11}$$

which is the one obtained by Shivakumara et al. [33]. When $C = 0$ (i.e. Newtonian fluid case), Eq. (3.11) reduces to

$$Ra_T^{St} = \frac{(a^2 + \pi^2)^2}{a^2} + \frac{(a^2 + \pi^2)}{a^2}\pi^2 Ta, \tag{3.12}$$

this coincides with the results of Vadasz [21]. Further $Ta = 0$, Eq. (3.12) gives

$$Ra_T^{st} = \frac{(\pi^2 + a^2)^2}{a^2}, \tag{3.13}$$

which has the critical value $Ra_c^{St} = 4\pi^2$ for $a_c^{St} = \pi^2$ obtained by Horton and Rogers [34] and Lapwood [35].

3.2. Oscillatory state

We now set $\sigma = i\sigma_i$ in Eq. (3.7) and clear the complex quantities from the denominator, to obtain

$$Ra_T = \Delta_1 + i\sigma_i\Delta_2, \tag{3.14}$$

where,

$$\Delta_1 = \frac{\delta^2}{a^2}\left(\eta\delta^2 - \frac{\sigma^2}{Pr_D}\right) + \frac{Ra_s(\delta^4 Le^{-1} + \sigma^2\chi)}{(\delta^2 Le^{-1})^2 + \sigma^2\chi^2} + \frac{\pi^2 Ta\left(\sigma^2 Pr_D^{-1} + \delta^2\eta\right)}{a^2\left(\eta^2 + \left(\sigma Pr_D^{-1}\right)^2\right)},$$

$$\Delta_2 = \frac{\delta^4}{a^2}\left(\eta + \frac{1}{Pr_D}\right) + \frac{Ra_s(\delta^2 Le^{-1} - \delta^2\chi)}{(\delta^2 Le^{-1})^2 + \sigma^2\chi^2} + \frac{\pi^2 Ta\left(\eta - \delta^2 Pr_D^{-1}\right)}{a^2\left(\eta^2 + \left(\sigma Pr_D^{-1}\right)^2\right)}.$$

Since Ra_T is a physical quantity, it must be real. Hence, from Eq. (3.14) it follows that either $\sigma_i = 0$ (steady onset) or $\Delta_2 = 0$ ($\sigma_i \neq 0$, oscillatory onset).

For oscillatory onset $\Delta_2 = 0$ ($\sigma_i \neq 0$) and this gives a dispersion relation of the form (on dropping the subscript i)

$$B_0(\sigma^2)^2 + B_1(\sigma^2) + B_2 = 0. \tag{3.15}$$

The expressions for the coefficients B_0, B_1 and B_2 are not presented here for brevity.

Now Eq. (3.14) with $\Delta_2 = 0$, gives

$$Ra_T^{Osc} = \frac{\delta^2}{a^2}\left(\eta\delta^2 - \frac{\sigma^2}{Pr_D}\right) + \frac{Ra_s(\delta^4 Le^{-1} + \sigma^2\chi)}{(\delta^2 Le^{-1})^2 + \sigma^2\chi^2} + \frac{\pi^2 Ta\left(\sigma^2 Pr_D^{-1} + \delta^2\eta\right)}{a^2\left(\eta^2 + \left(\sigma Pr_D^{-1}\right)^2\right)}. \tag{3.16}$$

We find the oscillatory neutral solutions from Eq. (3.16). It proceeds as follows: First determine the number of positive solutions of Eq. (3.15). If there are none, then no oscillatory instability is possible. If there are two, then the minimum (over a^2) of Eq. (3.16) with σ^2 given by Eq. (3.15) gives the oscillatory neutral Rayleigh number. Since Eq. (3.15) is quadratic in σ^2, it can give rise to more than one positive value of σ^2 for fixed values of the other parameters. The analytical expression for oscillatory Rayleigh number given by Eq. (3.16) is minimized with respect to the wavenumber numerically, after substituting for σ^2 from Eq. (3.15), for various values of physical parameters in order to know their effects on the onset of oscillatory convection.

4. Finite amplitude analysis with limited representation

In this section we consider the nonlinear analysis using a truncated representation of Fourier series considering two terms. Although the linear stability analysis is sufficient for obtaining the stability condition of the motionless solution and the corresponding eigenfunctions describing qualitatively the convective flow, it cannot provide information about the values of the convection amplitudes, nor regarding the rate of heat and mass transfer. To obtain this additional information, we perform the nonlinear analysis, which is useful to understand the physical mechanism with minimum amount of mathematical analysis and is a step forward towards understanding full nonlinear problem.

For simplicity of analysis, we confine ourselves to the two-dimensional rolls, so that all the physical quantities are independent of y. We introduce stream function ψ such that $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$ into the Eq. (2.2), eliminate pressure and non-dimensionalize the resulting equation and Eqs. (2.3) and (2.4) using the transformations (2.5) to obtain

$$\left(\frac{1}{Pr_D}\frac{\partial}{\partial t} + 1 - C\nabla^2\right)\nabla^2\psi - Ta^{1/2}\frac{\partial V}{\partial z} + Ra_T\frac{\partial T}{\partial x} - Ra_S\frac{\partial S}{\partial x} = 0, \tag{4.1}$$

$$\left(\frac{1}{Pr_D}\frac{\partial}{\partial t} + 1 - C\nabla^2\right)\frac{\partial V}{\partial z} + Ta^{1/2}\frac{\partial w}{\partial z} = 0, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T - \frac{\partial(\psi, T)}{\partial(x, z)} + \frac{\partial\psi}{\partial x} = 0, \tag{4.3}$$

$$\left(\chi\frac{\partial}{\partial t} - Le^{-1}\nabla^2\right)S - \frac{\partial(\psi, S)}{\partial(x, z)} + \frac{\partial\psi}{\partial x} = 0, \tag{4.4}$$

where V is the zonal velocity induced by rotation. A minimal double Fourier series which describes the finite amplitude convection is given by

$$\psi = A_1(t)\sin(ax)\sin(\pi z), \tag{4.5}$$

$$T = A_2(t)\cos(ax)\sin(\pi z) + A_3(t)\sin(2\pi z), \tag{4.6}$$

$$S = A_4(t)\cos(ax)\sin(\pi z) + A_5(t)\sin(2\pi z), \tag{4.7}$$

$$V = A_6(t)\sin(ax)\cos(\pi z) + A_7(t)\sin(2\pi x), \tag{4.8}$$

where the coefficients $A_1 - A_7$ are the time dependent amplitudes and are to be determined from the dynamics of the system. Substituting Eqs. (4.5)–(4.8) into Eqs. (4.1)–(4.4) and equating the coefficients of like terms we obtain the following non-linear autonomous system of differential equations

$$\frac{d\mathbf{X}}{dt} = \mathbf{H}, \tag{4.9}$$

where $\mathbf{X} = (\mathrm{A_i})^T$, $\mathbf{H} = (\mathrm{H_i})^T$ $i = 1 - 7$ with

$$H_1 = -\frac{Pr_D}{\delta^2}(\eta\delta^2 A_1 - \pi Ta^{1/2}A_6 + aRa_T A_2 - aRa_S A_4),$$

$$H_2 = -aA_1 - \delta^2 A_2 - \pi a A_1 A_3,$$

$$H_3 = -4\pi^2 A_3 + \frac{\pi a}{2}A_1 A_2,$$

$$H_4 = \frac{1}{\chi}(-aA_1 - \delta^2 Le^{-1}A_5 - \pi a A_1 A_6),$$

$$H_5 = \frac{1}{\chi}\left(-4\pi^2 Le^{-1}A_5 + \frac{\pi a}{2}A_1 A_4\right),$$

$$H_6 = -(Pr_D\eta A_6 + \pi Ta^{1/2}Pr_D A_1),$$

$$H_7 = -Pr_D\varpi A_7,$$

where $\varpi = 1 + 4C\pi^2$.

The non-linear system of autonomous differential equations is not suitable to analytical treatment for the general time-dependent variable and we have to solve it using a numerical method. However, one can make qualitative predictions as discussed below. The system of equations (4.9) is uniformly bounded in time and possesses many properties of the full problem. Like the original Eqs. (2.2)–(2.4), Eq. (4.9) must be dissipative. Thus volume in the phase space must contract. In order to prove volume contraction, we must show that velocity field has a constant negative divergence. Indeed,

$$\nabla \cdot U = \sum_{i=1}^{7}\frac{\partial}{\partial A_i}\left(\frac{dA_i}{dt}\right) = -[Pr_D(2\eta + \varpi) + (1 + Le^{-1})(\delta^2 + 4\pi^2)], \tag{4.10}$$

which is always negative and therefore the system is bounded and dissipative. As a result, the trajectories are attracted to a set of measure zero in the phase space; in particular they may be attracted to a fixed point, a limit cycle or, perhaps, a strange attractor. From Eq. (4.9) we conclude that if a set of initial points in phase space occupies a region $U(0)$ at time $t = 0$, then after some time t, the end points of the corresponding trajectories will fill a volume

$$U(t) = U(0)\exp[-(Pr_D(2\eta + \varpi) + (1 + Le^{-1})(\delta^2 + 4\pi^2))t]. \tag{4.11}$$

This expression indicates that the volume decreases exponentially with time. We can also infer that, the large Darcy–Prandtl number and very small Lewis number ($Le < 1$) tend to enhance dissipation. Finally, we note that the system of Eq. (4.9) are invariant under the transformation $(A_1, A_2, A_5, A_6, A_7) \to (-A_1, -A_2, -A_5, -A_6, -A_7)$.

4.1. Steady finite amplitude motions

From qualitative predictions we look into the possibility of an analytical solution. In the case of steady motions, Eqs. (4.1)–(4.4) can be solved in closed form. For steady state the system (4.9), after eliminating all amplitudes except A_1, gives

$$ph^2 + qh + r = 0, \tag{4.12}$$

where $h = \frac{A_1^2}{8}$, and $p = a^4Le^2(\delta^2\eta^2 - \pi^2 Ta)$,

$$q = a^2(-\pi^2 Ta\delta^2(1 + Le^2\delta^2) + \eta(a^2Le(Ra_S - LeRa_T\delta^2)) + \delta^4\eta(1 + Le^2)),$$
$$r = \delta^2(-\pi^2 Ta\delta^2 + \eta(a^2(LeRa_S - Ra_T)) + \delta^4\eta).$$

The required root of Eq. (4.12) is,

$$h = \frac{1}{2p}(-q + (q^2 - 4pr)^{1/2}). \tag{4.13}$$

When we let the radical in the above equation to vanish, we obtain the expression for finite amplitude Rayleigh number Ra_T^f, which characterizes the onset of finite amplitude steady motions. The finite amplitude Rayleigh number can be obtained in the form

$$Ra_T^f = \frac{1}{2x_1}\left(-x_2 + \left(x_2^2 - 4x_1x_3\right)^{1/2}\right), \tag{4.14}$$

where

$$x_1 = a^4Le^4\delta^4\eta^2,$$
$$x_2 = 2a^2Le^2\eta\delta^4\pi^2Ta(Le^2-1) - 2\eta^2a^4Le^3\delta^2Ra_S + 2\eta^3a^2Le^2\delta^6(1-Le^2),$$
$$x_3 = \pi^4Ta^2\delta^2(1+Le^4+Le^2(2-4\delta^2)) + 2\pi^2Ta\delta^2\eta(a^2LeRa_S(Le^2(-1+2\delta^2)-1) + \delta^4(4Le^2\delta^2-(1+Le^2)^2)\eta) + \eta^2a^4Le^2Ra_S^2 + 2a^2LeRa_S\delta^4(1+Le^2(1-2\delta^2))\eta^3 + \delta^8(1+Le^2(Le^2+2(1-2\delta^2)))\eta^4.$$

The expression for the steady finite amplitude Rayleigh number given by Eq. (4.14) is evaluated for critical values and the results are discussed in Section 5.

4.2. Heat and mass transports

In the study of convection in fluids, the quantification of heat and mass transport is important. This is because the onset of convection, as Rayleigh number is increased, is more readily detected by its effect on the heat and mass transport. In the basic state, heat and mass transport is by conduction alone.

If H and J are the rate of heat and mass transport per unit area respectively, then

$$H = -\kappa_T\left\langle\frac{\partial T_{total}}{\partial z}\right\rangle_{z=0} \quad \text{and} \quad J = -\kappa_S\left\langle\frac{\partial S_{total}}{\partial z}\right\rangle_{z=0}, \tag{4.15}$$

where the angular bracket corresponds to a horizontal average and

$$T_{total} = T_0 - \Delta T\frac{z}{d} + T(x,z,t) \quad \text{and} \quad S_{total} = S_0 - \Delta S\frac{z}{d} + S(x,z,t). \tag{4.16}$$

Substituting Eqs. (4.6) and (4.7) in Eq. (4.16) and using the resultant equations in Eq. (4.15), we get

$$H = \frac{\kappa_T\Delta T}{d}(1-2\pi A_3) \quad \text{and} \quad J = \frac{\kappa_S\Delta S}{d}(1-2\pi A_5). \tag{4.17}$$

The Nusselt number and Sherwood number are defined by

$$Nu = \frac{H}{\kappa_T\Delta T/d} = (1-2\pi A_3) \quad \text{and} \quad Sh = \frac{J}{\kappa_S\Delta S/d} = (1-2\pi A_5). \tag{4.18}$$

Writing A_3 and A_5 in terms of A_1, we obtain

$$Nu = 1 + \frac{2h}{h+\delta^2/a^2} \quad \text{and} \quad Sh = 1 + \frac{2h}{h+\delta^2/a^2Le^2}. \tag{4.19}$$

The second term on the right-hand side of Eq. (4.19) represent the convective contribution to heat and mass transport respectively.

5. Results and discussion

Both linear and weak nonlinear stability analysis of double diffusive convection in a rotating couple stress fluid-saturated porous medium is investigated. The linear theory is based on the usual normal mode technique and the nonlinear theory on the truncated Fourier series method. The expressions for the stationary, oscillatory, and finite amplitude Rayleigh numbers for different values of the parameters such as Taylor number, couple stress parameter, solute Rayleigh number, Lewis number, Darcy–Prandtl number, and normalized porosity are computed, and the results are depicted in figures.

The neutral stability curves in the (Ra_T,a) plane for various parameter values are as shown in Figs. 1–8. We fixed the values for the parameters as Ta = 20, C = 3,Ra_S = 100, Le = 20, Pr_D = 10 and χ = 0.4 except the varying parameter. From these figures it is clear that the neutral curves are connected in a topological sense. The effect of Taylor number Ta on the neutral curves is shown in Fig. 1. We find that for fixed values of all other parameters, the minimum value of the Rayleigh number for both stationary and oscillatory mode increases as a function of increasing Ta, indicating that the effect of Taylor number is to stabilize the system. Besides, the critical wave number increases (i.e., the size of convection cell decreases) with increasing Ta. Fig. 2 depicts the effect of couple stress parameter C on the neutral stability curves. We find that with an increase in the value of couple stress parameter, the minimum value of the Rayleigh number for both stationary and oscillatory mode increases indicating that it delays the onset of double diffusive convection.

The effect of solute Rayleigh number on the onset criteria is depicted in Fig. 3(a) and (b). We find that the minimum Rayleigh number for both stationary and oscillatory mode increases with an increase in the value of the solute Rayleigh number, indicating that the effect of solute Rayleigh number is to enhance the stability of the system. In Fig. 4(a) and (b) we display the effect of Lewis number Le on the neutral stability curves for fixed values of other parameters. It is observed that with the increase of Le the critical values of Rayleigh number and corresponding wavenumber for the overstable mode decrease while

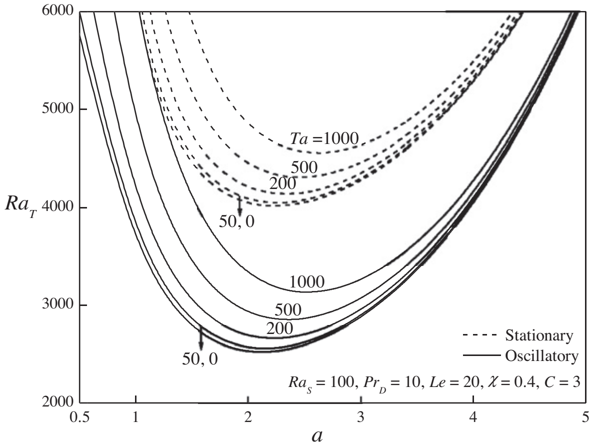

Fig. 1. Neutral stability curves for different values of Taylor number *Ta*.

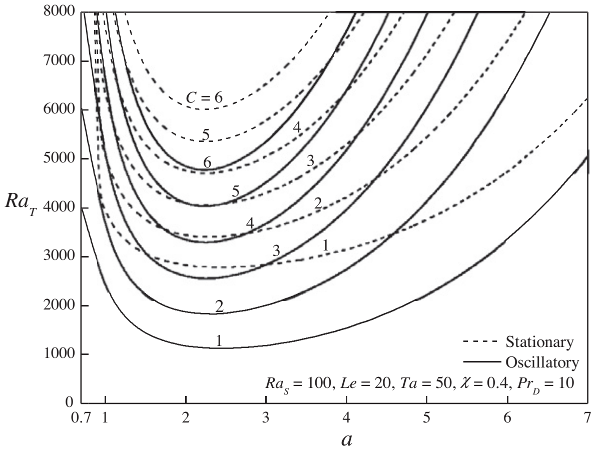

Fig. 2. Neutral stability curves for different values of couple stress parameter *C*.

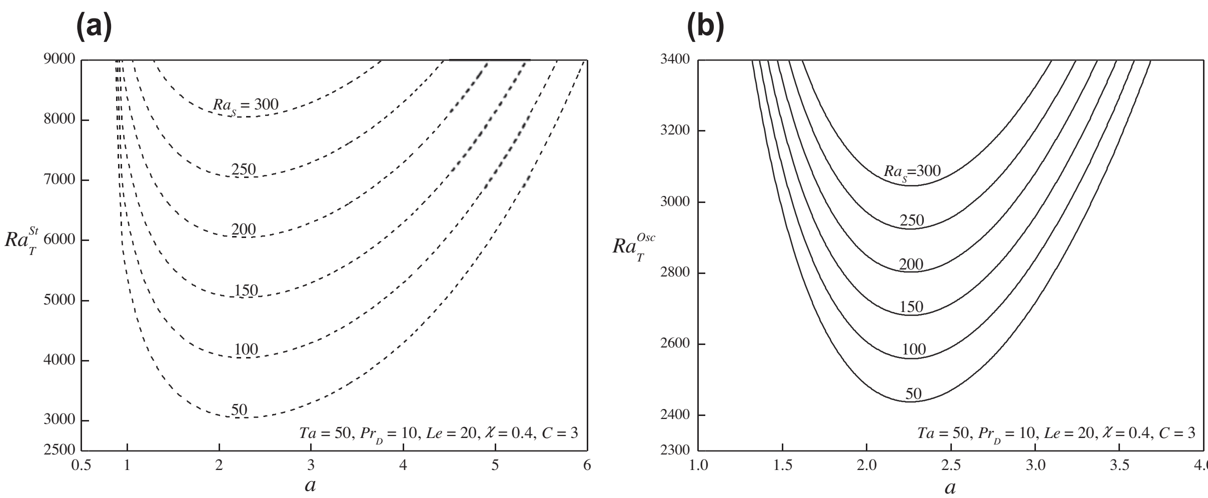

Fig. 3. (a) Stationary neutral stability curves for different values of solute Rayleigh number Ra_S. (b) Oscillatory neutral stability curves for different values of solute Rayleigh number Ra_S.

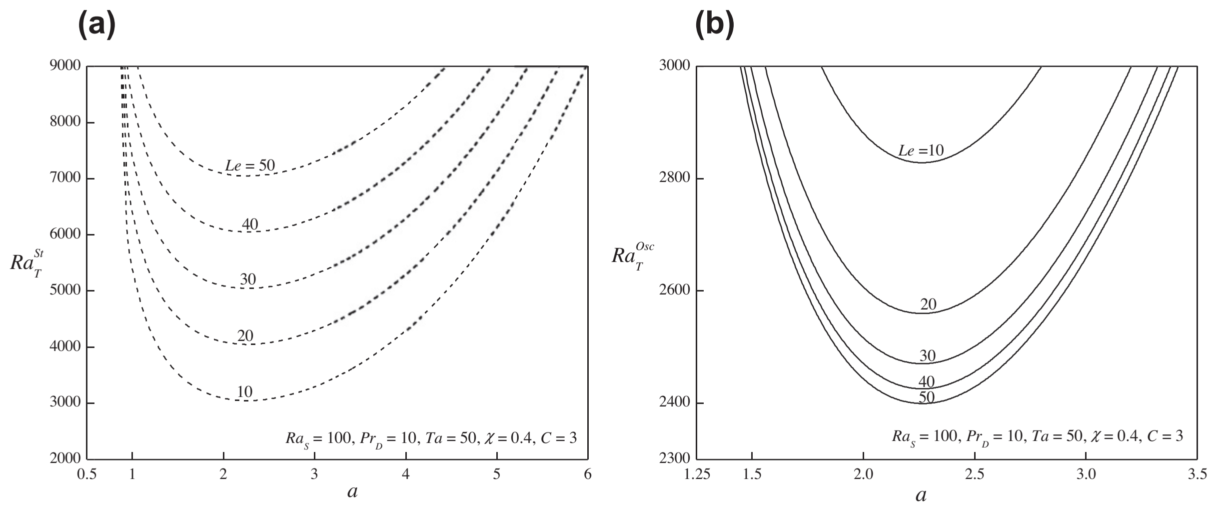

Fig. 4. (a) Stationary neutral stability curves for different values of Lewis number *Le*. (b) Oscillatory neutral stability curves for different values of Lewis number *Le*.

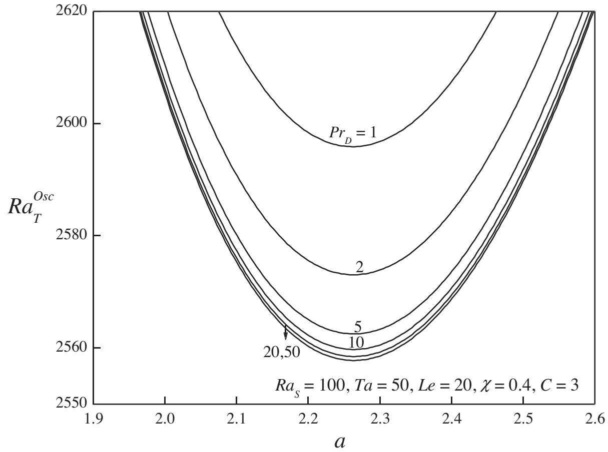

Fig. 5. Neutral stability curves for different values of Darcy–Prandtl number Pr_D.

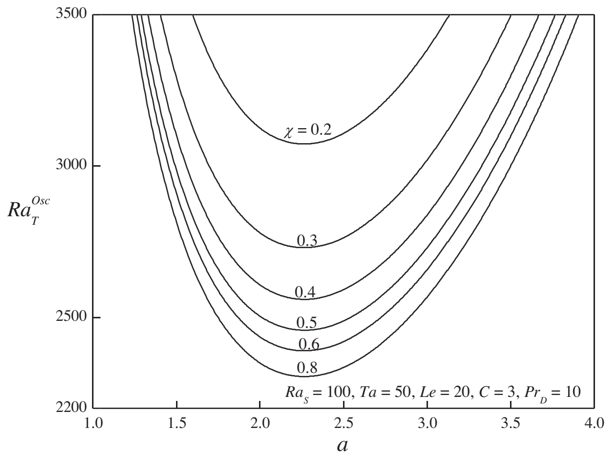

Fig. 6. Neutral stability curves for different values of normalized porosity χ.

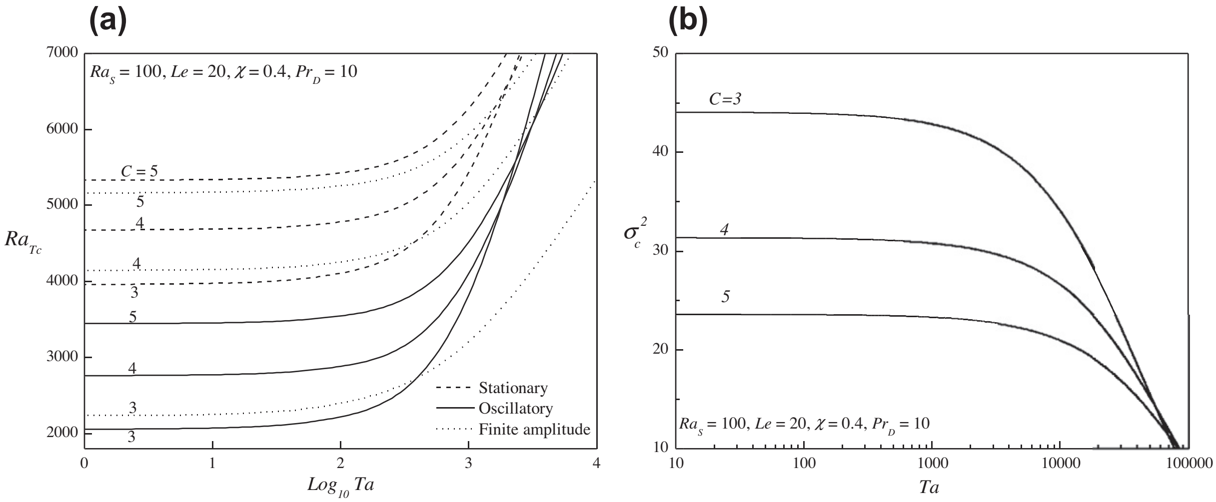

Fig. 7. (a) Variation of critical Rayleigh number with Taylor number *Ta* for different values of *C*. (b) Variation of critical frequency with Taylor number *Ta* for different values of *C*.

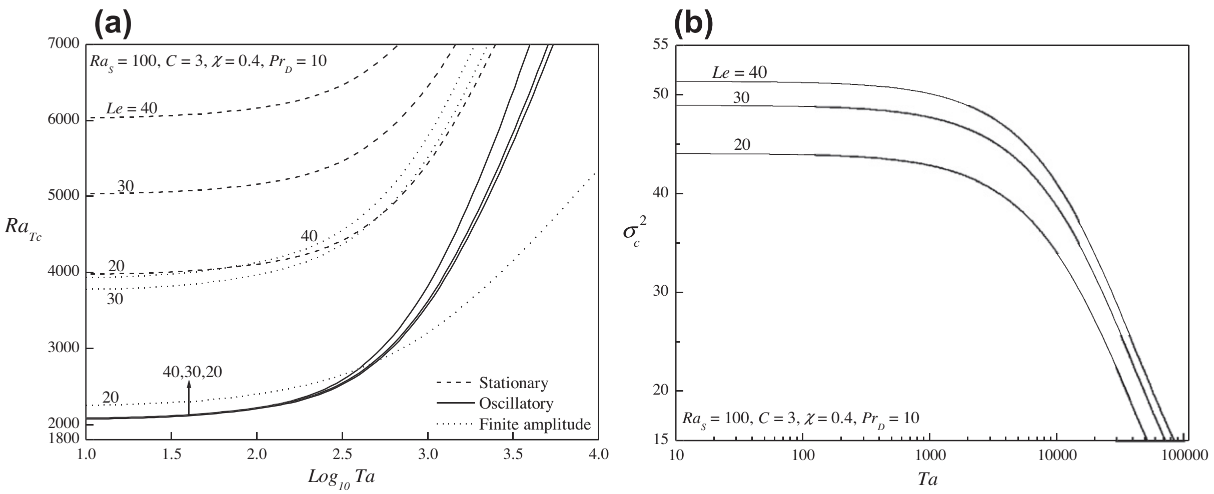

Fig. 8. (a) Variation of critical Rayleigh number with Taylor number *Ta* for different values of *Le*. (b) Variation of critical frequency with Taylor number *Ta* for different values of *Le*.

those for stationary mode increase. Therefore, the Lewis number has a contrasting effect on the stability of the system in stationary and oscillatory modes.

In Fig. 5 the marginal stability curves for different values of Darcy–Prandtl number Pr_D are drawn. We find that an increase in the value of the Darcy–Prandtl number Pr_D decreases the oscillatory Rayleigh number, indicating that the Darcy–Prandtl number advances the onset of oscillatory convection. The effect of normalized porosity parameter χ is depicted in the Fig. 6. We find that the effect of increasing the normalized porosity is to decrease the critical Rayleigh number for oscillatory mode, indicating that the effect of increasing χ is to advance the onset of oscillatory convection.

In Figs. 7–11, we show the critical Rayleigh number for stationary, oscillatory and finite amplitude modes and the critical frequency of the oscillatory mode as functions of the Taylor number for different values of all other parameters in order to know the preferred type of instability. We find that all of the quantities namely, the critical Rayleigh number for stationary, oscillatory and finite amplitude modes are increasing functions of the Taylor number. It is clear that for the parameters chosen for these figures the oscillatory convection sets in prior to the steady finite amplitude and stationary convection.

The variation of the critical Rayleigh number for stationary, oscillatory and finite amplitude modes and the critical frequency of oscillatory mode with Taylor number for different values of the couple stress parameter C as shown in Fig. 7(a) and (b). We observe that the critical Rayleigh number for stationary, oscillatory and finite amplitude mode increases with increasing couple stress parameter C when the Taylor number is fixed, indicating that couple stress parameter stabilizes

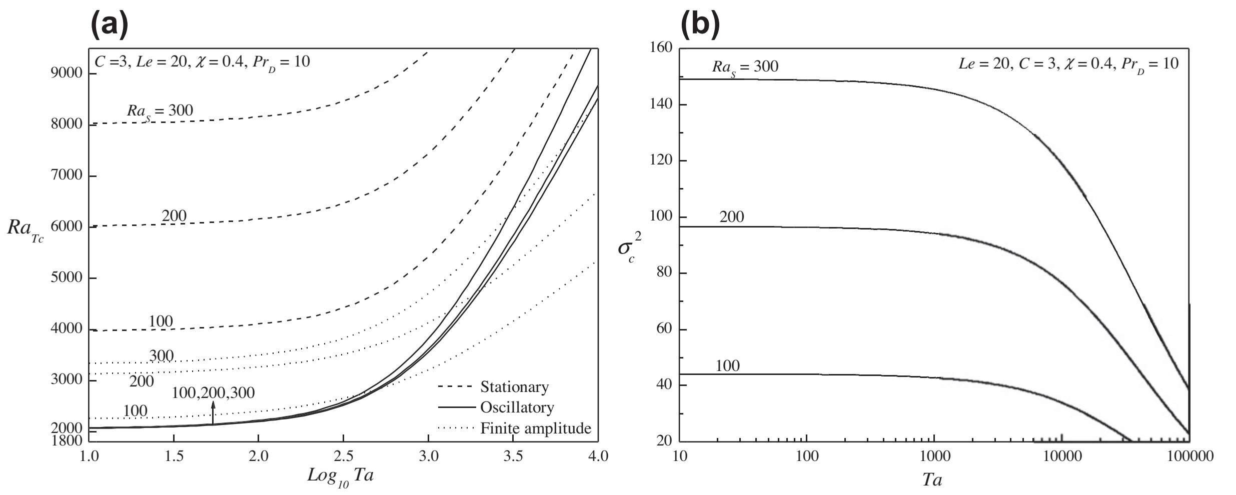

Fig. 9. (a) Variation of critical Rayleigh number with Taylor number Ta for different values of Ra_S. (b) Variation of critical frequency with Taylor number Ta for different values of Ra_n.

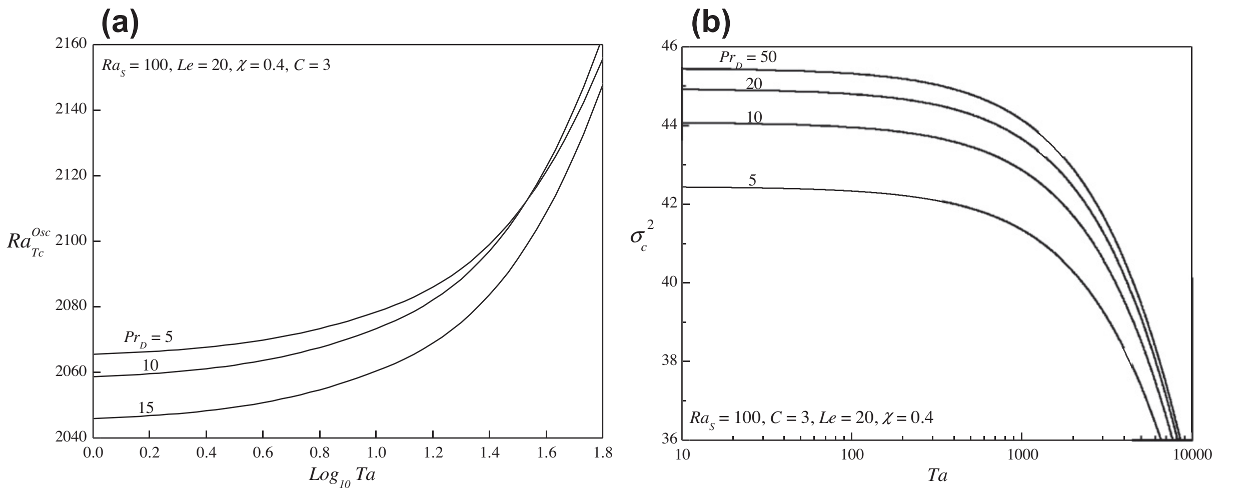

Fig. 10. (a) Variation of critical Rayleigh number with Taylor number Ta for different values of Pr_D. (b) Variation of critical frequency with Taylor number Ta for different values of Pr_D.

the system. The effect of couple stress parameter C on the critical frequency is displayed in Fig. 7(b). A marked decrease in the frequency with increasing couple stress parameter is noticed from the figure, while large values of the frequency are particularly related to small values of Ta, and they decay as Ta increases.

Fig. 8(a) and (b) shows the variation of the critical Rayleigh numbers for stationary, oscillatory and finite amplitude modes and the critical frequency of oscillatory mode with Taylor number for different values of the Lewis number Le. We find that the critical stationary and finite amplitude Rayleigh numbers increases with increase in the value of the Lewis number, indicating that the effect of Lewis number is to stabilize the system. On the other hand, increasing of Lewis number decreases the critical oscillatory Rayleigh number, indicating that the Lewis number destabilizes the system in oscillatory mode. This is because when $Le > 1$ the diffusivity of heat is more than the diffusivity of solute and therefore, solute gradient augments the onset of oscillatory convection. The result reflects the fact that the cause of oscillatory instability is the difference in the rates of diffusion of heat and solute. The effect of Lewis number Le on the critical frequency is revealed in Fig. 8(b). It is found that the critical frequency increases with increase in the value of Le.

The variation of critical stationary, oscillatory and finite amplitude Rayleigh numbers and critical frequency of oscillatory mode with Taylor number for different values of the solute Rayleigh number is shown in Fig. 9(a) and (b). We observe that the critical Rayleigh number for stationary, oscillatory and finite amplitude modes increases with increasing solute Rayleigh number, indicating that the solute Rayleigh number enhances the stability of the system. Fig. 9(b) shows that the critical frequency increases with increasing solute Rayleigh number.

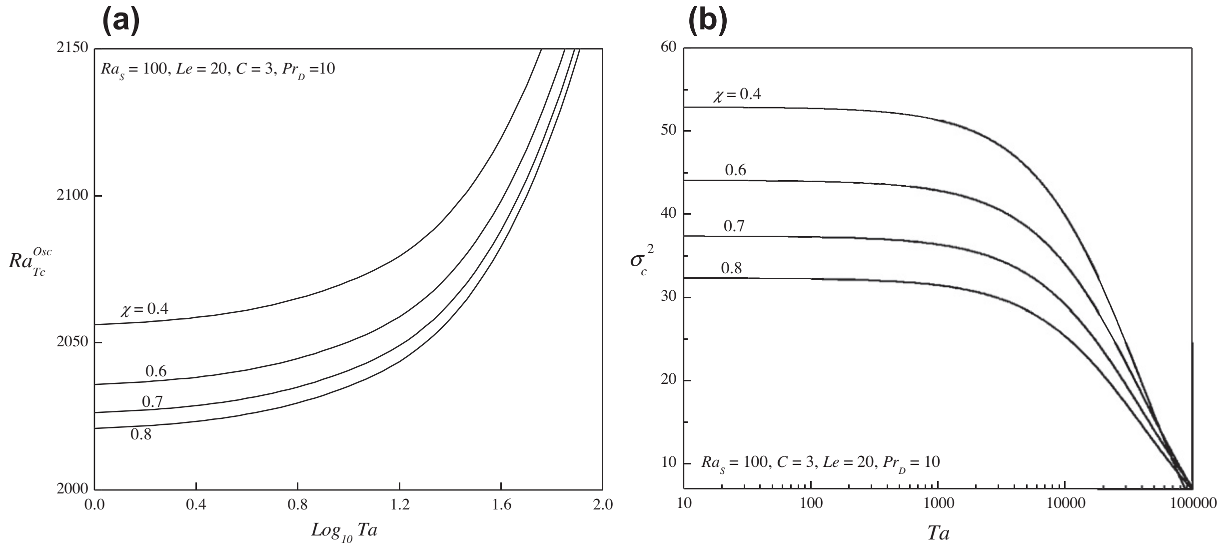

Fig. 11. (a) Variation of critical Rayleigh number with Taylor number *Ta* for different values of χ. (b) Variation of critical frequency with Taylor number *Ta* for different values of χ.

Fig. 10(a) and (b) shows the variation of critical Rayleigh number for oscillatory mode and the critical frequency with Taylor number for different values of the Darcy–Prandtl number. We observe that the critical oscillatory Rayleigh number decreases with an increase in the value of Darcy–Prandtl number, indicating that the Darcy–Prandtl number advances the onset of oscillatory convection. Further, the critical frequency increases with increasing Pr_D (Fig. 10(b)).

The effect of normalized porosity on the critical oscillatory Rayleigh number and critical frequency is displayed in Fig. 11(a) and (b). We find that the effect of increasing the normalized porosity is to decrease the critical Rayleigh number for oscillatory mode. As normalized porosity parameter increases, the thermal 'lag' effect (double-advective behavior in the terminology of Phillips [36]) is reduced. This makes advective heat transfer more effective and so makes it easier for the destabilizing thermal buoyancy gradient to produce convection. The critical frequency decreases with the normalized porosity (Fig. 11(b)).

In the study of double diffusive convection the determination of heat and mass transport across the layer plays a very important role. Here, the onset of convection as the Rayleigh number is increased is more rapidly detected by its effect on the heat and mass transfer. The quantity of heat and mass transfer across the layer are given by *Nu* and *Sh* respectively, which represent the ratio of heat or mass transported across the layer to the heat or mass transported by conduction alone. Figs. 12–15 indicates the effect of various parameters on the Nusselt number *Nu* and Sherwood number *Sh*.

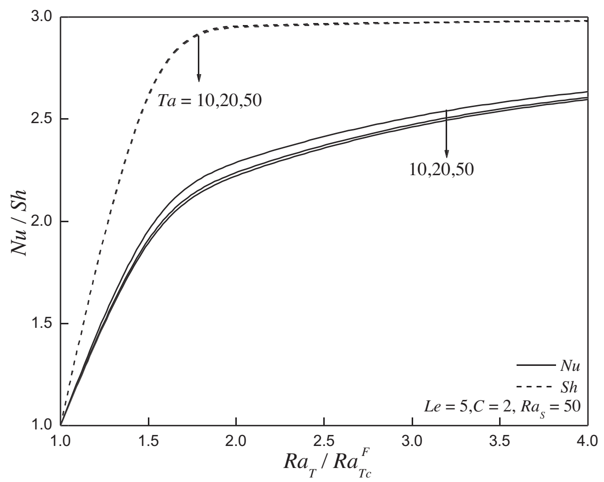

Fig. 12. Variation of Nusselt number and Sherwood number with critical Rayleigh number for different values of *Ta*.

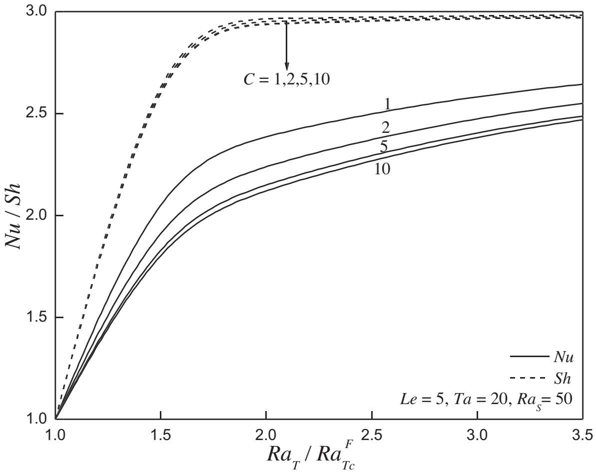

Fig. 13. Variation of Nusselt number and Sherwood number with critical Rayleigh number for different values of *C*.

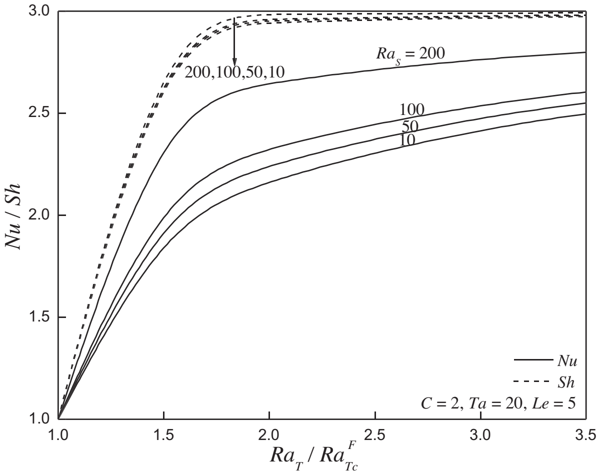

Fig. 14. Variation of Nusselt number and Sherwood number with critical Rayleigh number for different values of Ra_S.

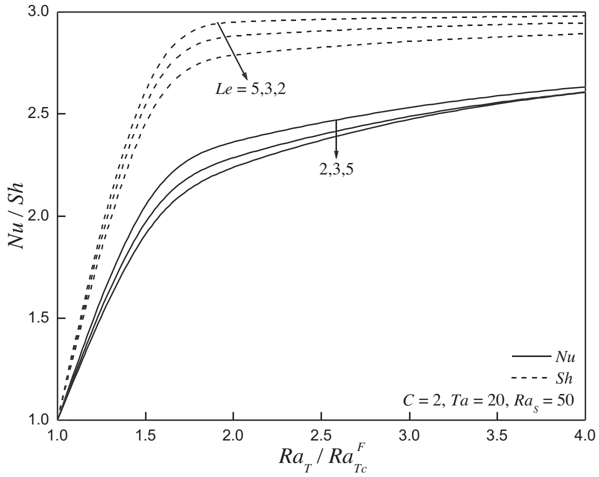

Fig. 15. Variation of Nusselt number and Sherwood number with critical Rayleigh number for different values of *Le*.

In each of these cases we observe that as Rayleigh number increases from one to four times of its critical value, the heat and mass transfer increase sharply and as Rayleigh number is increased further, they remain almost constant. It is also found that in each case the Sherwood number is above the Nusselt number. We note that the effect of increasing the Taylor number *Ta* and couple stress parameter *C* is to decrease the values of *Nu* and *Sh* (see Figs. 12 and 13) whereas that of Ra_S is to increase both *Nu* and *Sh* (see Fig. 14). Although the presence of a stabilizing gradient of solute will inhibit the onset of convection, due to the strong finite amplitude motions, which exist for large Rayleigh numbers, tend to mix the solute and redistribute it so that the interior layers of the fluid are more neutrally stratified. As a consequence of that the inhibiting effect of solute gradient is greatly reduced and hence fluid will convict more and more heat and mass when Ra_S is increased. Further the effect of increasing Lewis number *Le* is to decrease *Nu* whereas it increases *Sh* (see Fig. 15).

The autonomous system of unsteady finite amplitude equations is solved numerically using the Runge–Kutta method with suitable initial conditions. Then *Nu* and *Sh* are evaluated as a function of time *t*. The unsteady transient behaviour of *Nu* and *Sh* is shown graphically through Fig. 16(a) and (b). It is found that both *Nu* and *Sh* start with a conduction state value (i.e.,1) at $t = 0$ and then oscillate periodically about their steady state value (i.e., close to 3) for $t > 0$. This periodic variation of *Nu* and *Sh* is very short lived and decays as time progresses. The values of *Nu* and *Sh* then tend towards their steady state value 3. We find from Fig. 16(a) that increase in the value of Taylor number is to decrease the quantity of heat and mass transfer due to their stabilizing effect. From Fig. 16(b), it is clear that the heat and mass transfer increases with increasing couple stress parameter.

6. Conclusions

The onset of double diffusive convection in a couple stress fluid-saturated with rotating porous layer is studied using linear and weak nonlinear stability analyses. The modified Darcy equation that includes the time derivative term and the Coriolis term is used to model the momentum equation. The expressions for stationary, oscillatory, and finite amplitude Rayleigh number are obtained as a function of the governing parameters. The effect of the Taylor number, couples stress parameter, solute Rayleigh number, Darcy–Prandtl number, Lewis number, and normalized porosity on the stationary, oscillatory, and

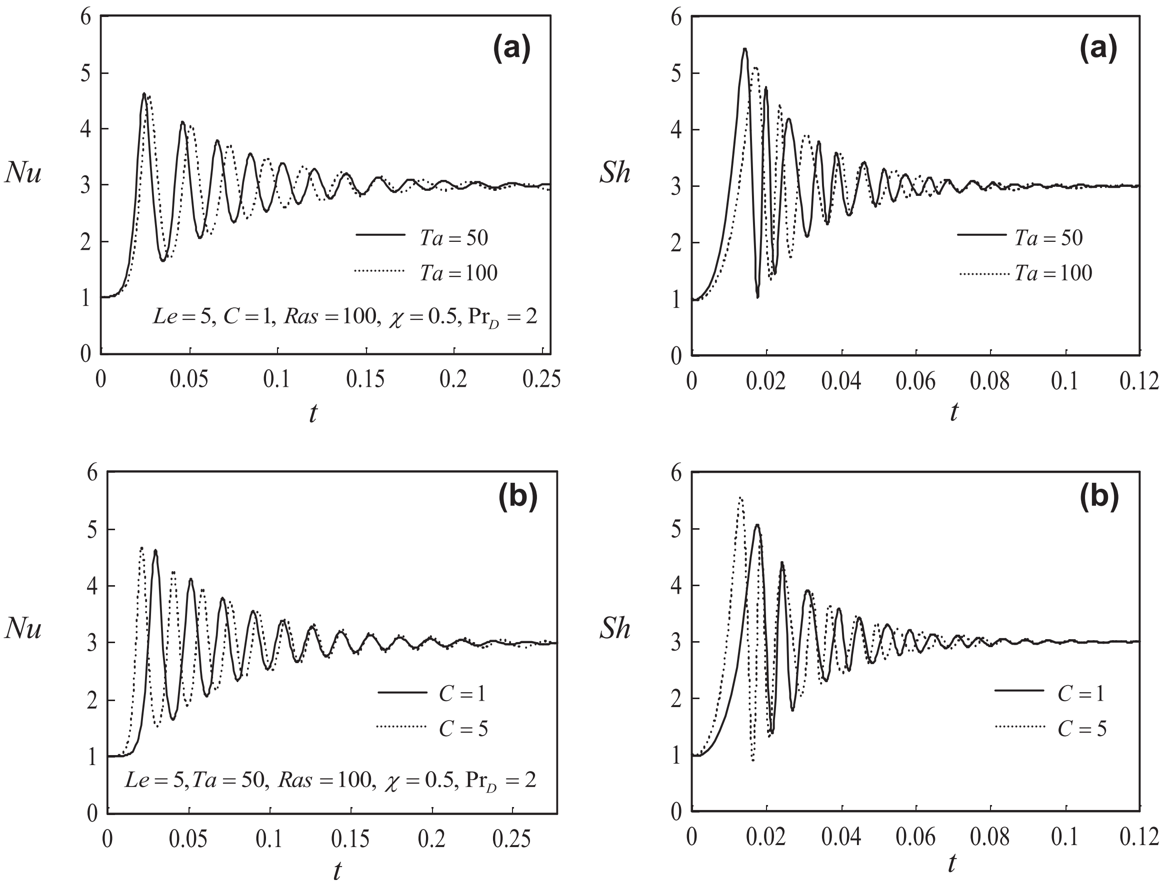

Fig. 16. Transient Nusselt number and sherwood number with time for different values of (a) *Ta* and (b) *C*.

finite amplitude convection are shown graphically. The critical Rayleigh number for stationary, oscillatory, and finite amplitude modes are increasing functions of the Taylor number. The oscillatory mode is the preferred mode of instability. The Taylor number, couple stress parameter and solute Rayleigh number have a stabilizing effect on the stationary, oscillatory, and finite amplitude convection. The Lewis number has stabilizing effect in the case of stationary and finite amplitude modes while it has a destabilizing effect in the case of oscillatory mode. The effect of increasing Darcy–Prandtl number and normalized porosity is to advances the onset of oscillatory convection which indicates the destabilizing effect. The heat and mass transfer decreases with an increase in the values of Taylor number Ta and couple stress parameter C, while both increases with an increase in the value of solute Rayleigh number Ra_S, and Lewis number Le reduces the heat transport while mass transport is reinforced. The transient Nusselt number and the Sherwood number approaches the steady state values at large time.

Acknowledgements

This work is supported by UGC New Delhi, under the Special Assistance Programme DRS-II.

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